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The asymptotic stability of $x_{n+1}-x_n+Ax_{n-k}=0$

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1. Introduction and main results

The asymptotic stability of delay difference equations has been investigated by many authers. In scalar case, many results can be found in several books and papers [1-5]. Levin *et.al* [5] and Kuruklis [4] have shown the nice result as follows:
The delay difference equation

$$x_{n+1}-x_n+qx_{n-k}=0, \quad n=0,1,\cdots,$$

where q is a real number and k is a nonnegative integer, is asymptotically stable if and only if

$$0<q<2\cos\frac{k\pi}{2k+1}.$$

In this paper we give some new necessary and sufficient conditions for the asymptotic stability of a 2-dimensional linear delay difference system

$$x_{n+1}-x_n+Ax_{n-k}=0, \quad n=0,1,\cdots, \tag{1}$$

where k is a nonnegative integer and A is a 2×2 constant matrix.

By the transformation $x_n=Py_n$ with an appropriate regular matrix P, we can rewrite (1) as

$$y_{n+1}-y_n+P^{-1}APy_{n-k}=0, \quad n=0,1,\cdots.$$

Thus, we only have to consider (1) where the matrix A is either of the folloing two matrices :

$$\text{(I)}\ A=qR(\theta)\equiv q\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, \qquad \text{(II)}\ A=\begin{pmatrix}q_1 & b\\ 0 & q_2\end{pmatrix},$$

where q, q_1, q_2, b and θ ($|\theta|\le\frac{\pi}{2}$) are real numbers.

For the case (I), we have

Theorem 1. *The system (1) is asymptotically stable if and only if*

$$0 < q < 2\cos\frac{k\pi + |\theta|}{2k+1}. \tag{2}$$

For the case (II), we have

Theorem 2. *The system (1) is asymptotically stable if and only if*

$$0 < q_1 < 2\cos\frac{k\pi}{2k+1} \quad \textit{and} \quad 0 < q_2 < 2\cos\frac{k\pi}{2k+1}. \tag{3}$$

In this paper we only give the proof of the Theorem 1 since both theorems are proved in very similar way.

2. Proof of Theorem 1

Theorem 1 is proved by using the fact that the system (1) is asymptotically stable if and only if all the roots of its characteristic equation

$$F(\lambda) \equiv \det(\lambda^{k+1}I - \lambda^k I + qR(\theta)) = 0 \tag{4}$$

are inside the unit disk. Hence, we investigate the characteristic roots of (4) to prove Theorem 1.

Let

$$f^+(\lambda) \equiv \lambda^{k+1} - \lambda^k + qe^{i\theta}, \quad f^-(\lambda) \equiv \lambda^{k+1} - \lambda^k + qe^{-i\theta}.$$

Then we have

$$\begin{aligned}
F(\lambda) &= \det\begin{pmatrix} \lambda^{k+1} - \lambda^k + q\cos\theta & -q\sin\theta \\ q\sin\theta & \lambda^{k+1} - \lambda^k + q\cos\theta \end{pmatrix} \\
&= (\lambda^{k+1} - \lambda^k + q\cos\theta)^2 + q^2\sin^2\theta \\
&= (\lambda^{k+1} - \lambda^k + q\cos\theta + iq\sin\theta)(\lambda^{k+1} - \lambda^k + q\cos\theta - iq\sin\theta) \\
&= f^+(\lambda)f^-(\lambda).
\end{aligned}$$

Note that $f^-(\lambda) = 0$ implies $f^+(\bar{\lambda}) = 0$, where $\bar{\lambda}$ is the complex conjugate of any complex λ. Also, when $-\frac{\pi}{2} \le \theta \le 0$, substituting $\tilde{\theta} = -\theta$ in $f^+(\lambda) = 0$ and $f^-(\lambda) = 0$ implies $0 \le \tilde{\theta} \le \frac{\pi}{2}$. Therefore, we only have to consider the case $f^+(\lambda) = 0$ under the condition $0 \le \theta \le \frac{\pi}{2}$ to investigate the characteristic roots of (4). We also notice that (4) has no real roots if $q \ne 0$ and $\theta \ne 0$.

When $k = 0$, it follows from (4) that $\lambda = 1 - q\cos\theta \pm iq\sin\theta$. Then

$$|\lambda|^2 = (1 - q\cos\theta)^2 + (q\sin\theta)^2 = q^2 - 2q\cos\theta + 1.$$

It is easy to see that $|\lambda| < 1$ if and only if $0 < q < 2\cos\theta$, and so, Theorem 1 stands if $k = 0$. Hereafter, let k be a positive integer.

As a beginning, we shall examine the existence region of the arguments of complex roots of $f^+(\lambda) = 0$.

Lemma 1. *Assume that* $q > 0$ *and* $0 \le \theta \le \frac{\pi}{2}$. *Let* $re^{i\omega}$, *with* $r > 0$ *and* $0 < |\omega| < \pi$, *be a complex root of* $f^+(\lambda) = 0$. *Then*

$$\frac{\theta + 2p\pi}{k} < \omega < \frac{\theta + (2p+1)\pi}{k+1}, \quad p = 0, 1, \cdots, \left[\frac{k-1}{2}\right], \text{ if } \omega > 0;$$

$$\frac{\theta - \pi}{k+1} < \omega < 0, \quad \frac{\theta - (2p+1)\pi}{k+1} < \omega < \frac{\theta - 2p\pi}{k}, \quad p = 1, 2, \cdots, \left[\frac{k}{2}\right], \text{ if } \omega < 0,$$

where $[a]$ *represents the integer part of* a.

Proof. From $f^+(re^{i\omega}) = 0$, we have $r^{k+1}e^{i((k+1)\omega-\theta)} - r^k e^{i(k\omega-\theta)} + q = 0$, namely

$$r^k\{\cos(k\omega - \theta) - r\cos((k+1)\omega - \theta)\} = q, \tag{5}$$

and

$$r^k\{\sin(k\omega - \theta) - r\sin((k+1)\omega - \theta)\} = 0. \tag{6}$$

It is obvious that $\sin((k+1)\omega - \theta) \neq 0$, so (6) implies

$$r = \frac{\sin(k\omega - \theta)}{\sin((k+1)\omega - \theta)}. \tag{7}$$

(5) and (7) yield

$$\begin{aligned} q &= r^k \frac{\sin((k+1)\omega - \theta)\cos(k\omega - \theta) - \sin(k\omega - \theta)\cos((k+1)\omega - \theta)}{\sin((k+1)\omega - \theta)} \\ &= r^k \frac{\sin\omega}{\sin((k+1)\omega - \theta)}. \end{aligned} \tag{8}$$

We consider the case $0 < \omega < \pi$. (In case $-\pi < \omega < 0$, the proof is similar.) From (7) and (8), we must have $\sin(k\omega - \theta) > 0$ and $\sin((k+1)\omega - \theta) > 0$ because of $q > 0, r > 0$ and $\sin\omega > 0$. Hence,

$$\frac{\theta + 2m\pi}{k} < \omega < \frac{\theta + (2m+1)\pi}{k}, \quad m = 0, 1, \cdots, \left[\frac{k-1}{2}\right],$$

and

$$\frac{\theta + 2n\pi}{k+1} < \omega < \frac{\theta + (2n+1)\pi}{k+1}, \quad n = 0, 1, \cdots, \left[\frac{k}{2}\right],$$

which imply

$$\frac{\theta + 2p\pi}{k} < \omega < \frac{\theta + (2p+1)\pi}{k+1}, \quad p = 0, 1, \cdots, \left[\frac{k-1}{2}\right].$$

The proof is complete.

When $q < 0$, we have the following analogous result.

Lemma 2. *Assume that $q < 0$ and $0 \le \theta \le \frac{\pi}{2}$. Let $re^{i\omega}$, with $r > 0$ and $0 < |\omega| < \pi$, be a complex root of $f^+(\lambda) = 0$. Then*

$$0 < \omega < \frac{\theta}{k+1}, \quad \frac{\theta + (2p-1)\pi}{k} < \omega < \frac{\theta + 2p\pi}{k+1}, \quad p = 1, 2, \cdots, \left[\frac{k}{2}\right], \text{ if } \omega > 0;$$

$$\frac{\theta - (2p+2)\pi}{k+1} < \omega < \frac{\theta - (2p+1)\pi}{k}, \quad p = 0, 1, \cdots, \left[\frac{k-1}{2}\right], \text{ if } \omega < 0.$$

The next lemma determines the value of q and the root's argument ω on the unit circle.

Lemma 3. *Assume that $q > 0$ and $0 \le \theta \le \frac{\pi}{2}$. Then the arguments of complex roots of $f^+(\lambda) = 0$ on the unit circle are given by ω_p^+ or ω_p^-, where*

$$\omega_p^+ \equiv \frac{2\theta + (4p+1)\pi}{2k+1} > 0, \qquad p = 0, 1, \cdots, \left[\frac{k-1}{2}\right],$$

$$\omega_p^- \equiv \frac{2\theta - (4p+1)\pi}{2k+1} < 0, \qquad p = 0, 1, \cdots, \left[\frac{k}{2}\right].$$

Moreover, the following relation stands :

$$q = \sqrt{2 - 2\cos\omega}. \tag{9}$$

Proof. Substituting $r = 1$ into (5) and (6), we get

$$\cos(k\omega - \theta) - \cos((k+1)\omega - \theta) = q, \tag{10}$$

and

$$\sin(k\omega - \theta) - \sin((k+1)\omega - \theta) = 0. \tag{11}$$

(11) implies that $2\cos\frac{(2k+1)\omega - 2\theta}{2}\sin\frac{\omega}{2} = 0$. Since $\sin\frac{\omega}{2} \ne 0$, we have

$$\cos\frac{(2k+1)\omega - 2\theta}{2} = 0. \tag{12}$$

We consider the case $0 < \omega < \pi$. (In case $-\pi < \omega < 0$, the proof is similar.) Then (12) yields

$$\omega = \frac{2\theta + (2n+1)\pi}{2k+1}, \quad n = 0, 1, \cdots, k-1. \tag{13}$$

By Lemma 1, (13) is suitable when n is only even, and therefore, we obtain

$$\omega = \omega_p^+ \equiv \frac{2\theta + (4p+1)\pi}{2k+1}, \quad p = 0, 1, \cdots, \left[\frac{k-1}{2}\right].$$

Next, by squaring both sides of (10) and (11), and adding them together, we have

$$\begin{aligned} q^2 &= \{\cos(k\omega-\theta)-\cos((k+1)\omega-\theta)\}^2+\{\sin(k\omega-\theta)-\sin((k+1)\omega-\theta)\}^2 \\ &= 2-2\{\cos((k+1)\omega-\theta)\cos(k\omega-\theta)+\sin((k+1)\omega-\theta)\sin(k\omega-\theta)\} \\ &= 2-2\cos\omega. \end{aligned}$$

Hence, (9) stands. The proof is complete.

Remark 1. In view of the definitions of ω_p^+ and ω_p^-, the value of $q(\omega)$ given by (9) is minimum when $\omega=\omega_0^-$. Taking account of $\sqrt{2-2\cos\omega}=2|\sin\frac{\omega}{2}|$, we obtain that

$$q(\omega_0^-)=2\sin\left|\frac{2\theta-\pi}{2(2k+1)}\right|=2\sin\left(\frac{\pi}{2}-\frac{k\pi+\theta}{2k+1}\right)=2\cos\frac{k\pi+\theta}{2k+1}.$$

When $q<0$, we have the following result which is analogous to Lemma 3.

Lemma 4. *Assume that $q<0$ and $0\le\theta\le\frac{\pi}{2}$. Then the arguments of complex roots of $f^+(\lambda)=0$ on the unit circle are given by α_p^+ or α_p^-, where*

$$\alpha_p^+\equiv\frac{2\theta+(4p-1)\pi}{2k+1}>0,\qquad p=1,2,\cdots,\left[\frac{k}{2}\right],$$
$$\alpha_p^-\equiv\frac{2\theta-(4p+3)\pi}{2k+1}<0,\qquad p=0,1,\cdots,\left[\frac{k-1}{2}\right].$$

Furthermore, we shall observe the crossing of the unit circle by the roots of $f^+(\lambda)=0$ when the value of q varies.

Lemma 5. *Assume that $0\le\theta<\frac{\pi}{2}$. Then the simple root $\lambda=1$ of $f^+(\lambda)=0$ with $q=0$ moves inside the unit disk (resp. outside the unit disk) as q increases from 0 (resp. decreases from 0).*

Proof. It suffices to show that $(dr/dq)\big|_{\substack{r=1\\q=0}}<0$. If $\theta=0$, let $\lambda=r$ be a positive root of $f^+(\lambda)=0$, then

$$r^{k+1}-r^k+q=0. \tag{14}$$

Note that $r=1$ implies $q=0$. Taking the derivative of r with q on (14), we have

$$(k+1)r^k\frac{dr}{dq}-kr^{k-1}\frac{dr}{dq}+1=0,$$

or

$$\frac{dr}{dq}=\frac{1}{kr^{k-1}-(k+1)r^k}.$$

Hence, we arrive at

$$\left.\frac{dr}{dq}\right|_{r=1} = \frac{1}{k-(k+1)} = -1 < 0.$$

If $0 < \theta < \frac{\pi}{2}$, it follows from (8) that

$$\begin{aligned}
\frac{dr}{dq} &= \frac{dr}{d\omega}\left(\frac{dq}{d\omega}\right)^{-1} \\
&= \frac{dr}{d\omega}\left(kr^{k-1}\frac{dr}{d\omega}\frac{\sin\omega}{\sin((k+1)\omega-\theta)} + r^k\frac{d}{d\omega}\left(\frac{\sin\omega}{\sin((k+1)\omega-\theta)}\right)\right)^{-1} \\
&= \frac{dr}{d\omega}\left(\frac{qk}{r}\frac{dr}{d\omega} + r^k\frac{d}{d\omega}\left(\frac{\sin\omega}{\sin((k+1)\omega-\theta)}\right)\right)^{-1}. \qquad (15)
\end{aligned}$$

Also, by (7), we have

$$\begin{aligned}
\frac{dr}{d\omega} &= \frac{k\cos(k\omega-\theta)\sin((k+1)\omega-\theta)-(k+1)\sin(k\omega-\theta)\cos((k+1)\omega-\theta)}{\sin^2((k+1)\omega-\theta)} \\
&= \frac{k\sin\omega-\sin(k\omega-\theta)\cos((k+1)\omega-\theta)}{\sin^2((k+1)\omega-\theta)},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{d\omega}\left(\frac{\sin\omega}{\sin((k+1)\omega-\theta)}\right) \\
&\quad = \frac{\cos\omega\sin((k+1)\omega-\theta)-(k+1)\sin\omega\cos((k+1)\omega-\theta)}{\sin^2((k+1)\omega-\theta)} \\
&\quad = \frac{\sin(k\omega-\theta)-k\sin\omega\cos((k+1)\omega-\theta)}{\sin^2((k+1)\omega-\theta)}.
\end{aligned}$$

Denote

$$G(\omega) \equiv k\sin\omega - \sin(k\omega-\theta)\cos((k+1)\omega-\theta), \qquad (16)$$

and

$$H(\omega) \equiv \sin(k\omega-\theta) - k\sin\omega\cos((k+1)\omega-\theta) \qquad (17)$$

then (15) yields

$$\frac{dr}{dq} = \frac{G(\omega)}{(qk/r)G(\omega)+r^kH(\omega)}. \qquad (18)$$

Noticing that $q=0$ is equivalent to $\omega=0$, we obtain that

$$\left.\frac{dr}{dq}\right|_{\substack{r=1\\q=0}} = \frac{G(0)}{H(0)} = \frac{-\sin(-\theta)\cos(-\theta)}{\sin(-\theta)} = -\cos\theta < 0.$$

The proof is complete.

Lemma 6. *Assume that $q > 0$ and $0 \le \theta \le \frac{\pi}{2}$. Then all the roots of $f^+(\lambda) = 0$ on the unit circle move outside as q increases.*

Proof. By Lemma 3, it suffices to show that

$$\left.\frac{dr}{dq}\right|_{\substack{r=1 \\ q=q(\omega_p^+)}} > 0, \qquad p = 0, 1, \cdots, \left[\frac{k-1}{2}\right], \tag{19}$$

and

$$\left.\frac{dr}{dq}\right|_{\substack{r=1 \\ q=q(\omega_p^-)}} > 0, \qquad p = 0, 1, \cdots, \left[\frac{k}{2}\right]. \tag{20}$$

From (18), we have

$$\left.\frac{dr}{dq}\right|_{\substack{r=1 \\ q=q(\omega_p^+)}} = \frac{G(\omega_p^+)}{qkG(\omega_p^+) + H(\omega_p^+)},$$

where $G(\omega)$ and $H(\omega)$ are defined by (16) and (17) respectively. Note that $\sin \omega_p^+ > 0$ and $\sin(k\omega_p^+ - \theta) > 0$ because of Lemma 1. Also, since

$$\frac{2\theta + (4p+1)\pi}{2(k+1)} < \omega_p^+ < \frac{\theta + (2p+1)\pi}{k+1},$$

it is easy to see that $\cos((k+1)\omega_p^+ - \theta) < 0$. Thus, we obtain that $G(\omega_p^+) > 0$ and $H(\omega_p^+) > 0$, and so (19) holds. Similarly, we can show (20). The proof is complete.

When $q < 0$, using Lemmas 2 and 4 instead of Lemmas 1 and 3, we have the following result which is analogous to Lemma 6.

Lemma 7. *Assume that $q < 0$ and $0 \le \theta \le \frac{\pi}{2}$. Then all the roots of $f^+(\lambda) = 0$ on the unit circle move outside as $|q|$ increases.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let λ be a characteristic root of (4). Here, recalling the argument above, we only have to consider the value of λ satisfying $f^+(\lambda) = 0$ with $0 \le \theta \le \frac{\pi}{2}$.

(*Sufficiency*) From (2), we must have $0 \le \theta < \frac{\pi}{2}$. Thus, by virtue of Lemma 5 and continuity of λ with respect to q, we notice that, if $q > 0$ is sufficient small, then $|\lambda| < 1$ holds for any λ and the system (1) is asymptotically stable.

If the increasing of q leads the system (1) to instability, there exists a root λ^* of $f^+(\lambda) = 0$ such that $|\lambda^*| = 1$. By Remark 1, we find that $2\cos\frac{k\pi+\theta}{2k+1}$ is the minimum value of q when $|\lambda^*| = 1$. This fact indicates that, if (2) is true, then $|\lambda| < 1$ holds for any λ and the system (1) is asymptotically stable.

(*Necessity*) Suppose that the system (1) is asymptotically stable, that is, for any λ,

$$|\lambda| < 1. \tag{21}$$

For the sake of contradiction, (2) is false. Then we consider two cases.

Case 1: $q \geq 2\cos\frac{k\pi+\theta}{2k+1}$. By Lemma 6, there exists a root λ^* of $f^+(\lambda) = 0$ such that $|\lambda^*| \geq 1$ as q increases from $2\cos\frac{k\pi+\theta}{2k+1}$, which contradicts (21).

Case 2: $q \leq 0$. By Lemma 5, the simple root $\lambda = 1$ with $q = 0$ moves outside the unit disk as $|q|$ increases from 0. Hence, in view of the fact above and Lemma 7, there exists a root λ^* of $f^+(\lambda) = 0$ such that $|\lambda^*| \geq 1$, which also contradicts (21).

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